Fitting a near horizontal plane.

Ed Williams

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1 Introduction

Best-fitting a plane to an arbitrary collection \( x_i, y_i, z_i \) of \( N \) points in three space is typically done by minimizing the sum of the squares of the perpendicular distances of the points from the plane. This minimizes

\[
E_0 = \sum_i \left(\frac{(ax_i + by_i + cz_i)}{\sqrt{a^2 + b^2 + c^2}} - d\right)^2
\]

over the choices of \( \{a, b, c, d\} \) This is best done by singular value decomposition. However, if we take advantage of an assumption that the required plane is close to being one of constant \( z \), we can instead minimize

\[
E = \sum_i (z - z_i)^2 = \sum_i (ax_i + by_i + d - z_i)^2
\]

which is the sum of the squares of the height \( z \) deviations of the sample points from the fitting plane \( z = ax + by + d \). This is a much simpler problem that has an explicit solution requiring no iteration or linear algebra package. The minimum of \( E \) occurs when the partial derivatives of \( E \) vanish, that is:

\[
\frac{\partial E}{\partial a} = \frac{\partial E}{\partial b} = \frac{\partial E}{\partial d} = 0
\]

We can write these as:

\[
a \bar{x}^2 + b \bar{xy} + d \bar{x} - \bar{z} \bar{x} = 0 \tag{4}
\]

\[
a \bar{xy} + b \bar{y}^2 + d \bar{y} - \bar{z} \bar{y} = 0 \tag{5}
\]

\[
a \bar{x} + b \bar{y} + d - \bar{z} = 0 \tag{6}
\]

where the barred quantities are averages over the set of points, e. g.:

\[
\bar{x} \equiv \frac{\sum_i x_i}{N} \tag{7}
\]

\[
\bar{xy} \equiv \frac{\sum_i x_i y_i}{N} \tag{8}
\]

1
and so on.

Equation 6 tells us the centroid of our point set \( \{x, y, z\} \) lies on the best fit plane. We can thus simplify Eqs. 4 and 5 by rewriting them in terms of coordinates relative to the centroid, that is

\[
    X_i \equiv x_i - \bar{x} \quad (9)
\]
\[
    Y_i \equiv y_i - \bar{y} \quad (10)
\]
\[
    Z_i \equiv z_i - \bar{z} \quad (11)
\]

when they become

\[
    a \bar{X}^2 + b \bar{X} \bar{Y} = \bar{X} \bar{Z} \quad (12)
\]
\[
    a \bar{X} \bar{Y} + b \bar{Y}^2 = \bar{Y} \bar{Z} \quad (13)
\]

which we can readily solve for \( a \) and \( b \)

\[
    a = \frac{(\bar{X} \bar{Z} \bar{Y}^2 - \bar{Y} \bar{Z} \bar{X} \bar{Y})}{D} \quad (14)
\]
\[
    b = \frac{(\bar{Y} \bar{Z} \bar{X}^2 - \bar{X} \bar{Z} \bar{X} \bar{Y})}{D} \quad (15)
\]

\[
    D \equiv \bar{X}^2 \bar{Y}^2 - (\bar{X} \bar{Y})^2 \quad (16)
\]

\( d \) comes from Eq. 6:

\[
    d = \bar{z} - a \bar{x} - b \bar{y} \quad (17)
\]

giving us the parameters \( \{a, b, d\} \) of our least squares fit plane.

Note that \( a \) and \( b \) do not exist if \( D = 0 \), which is, in fact, the condition that we have at least 3 \( \{x_i, y_i\} \) points, and that they are not all collinear. These are the geometrical requirements that a best fit plane exists.

Algorithmically, first compute \( \bar{x}, \bar{y} \) and \( \bar{z} \) from the \( \{x_i, y_i, z_i\} \). Then replace the \( \{x_i, y_i, z_i\} \) with \( \{X_i, Y_i, Z_i\} \) using Eqs. 9-11.

Use these to compute the moments \( \{\bar{X}^2, \bar{Y}^2, \bar{X} \bar{Y}, \bar{X} \bar{Z}, \bar{Y} \bar{Z}\} \) by

\[
    \bar{X}^2 = \sum_i X_i^2/N \quad (18)
\]
\[
    \bar{Y}^2 = \sum_i Y_i^2/N \quad (19)
\]
\[
    \bar{X} \bar{Y} = \sum_i X_i Y_i/N \quad (20)
\]

etc.

Use Eq. 16 to compute \( D \). If it is near zero to rounding error, throw an error condition. Otherwise compute \( a, b \) and \( d \), the parameters of our best fit plane \( z = ax + by + d \), from Eqs. 14, 15 and 17 respectively.

This requires no iteration and minimal storage beyond that of the original point array.
2 Fitting an arbitrary plane

Finding the best fit (minimum of $E_0$) from Eq. 1 requires a more sophisticated approach. We use the method of Lagrange multipliers, instead minimizing $\sum_i (ax_i + by_i + cz_i - d)^2$ subject to the constraint $a^2 + b^2 + c^2 = 1$. Geometrically, $\{a, b, c\}$ is a unit normal to the plane, and $d$ is the distance to the plane from the origin.

We thus minimize the function

$$E_2 = \sum_i (ax_i + by_i + cz_i - d)^2 - \lambda(a^2 + b^2 + c^2) \quad (21)$$

with respect to $\{a, b, c, d\}$, obtaining

$$a\overline{x}^2 + b\overline{y} + c\overline{z} - d\overline{x} = \lambda a \quad (22)$$
$$a\overline{x}\overline{y} + b\overline{y}^2 + c\overline{y}\overline{z} - d\overline{y} = \lambda b \quad (23)$$
$$a\overline{x}\overline{z} + b\overline{y}\overline{z} + c\overline{z}^2 - d\overline{z} = \lambda c \quad (24)$$
$$a\overline{x} + b\overline{y} + c\overline{z} - d = 0 \quad (25)$$

As before, Eq. 25 tells us that the centroid lies on our fitting plane, so we again use coordinates relative to the centroid (Eqs. 4-6), obtaining

$$M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (26)$$

Note that the square error is given by

$$\sum_i (ax_i + by_i + cz_i - d)^2 = \sum_i (aX_i + bY_i + cZ_i)^2 = N (a \quad b \quad c) M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = N\lambda \quad (27)$$

$(a \quad b \quad c)$ is thus the normalized eigenvector of $M$ with the smallest eigenvalue. A linear algebra package, involving iteration is required to solve this. $d$ is then obtained from Eq. 25.