The method described here is based on ideas from [1].

1 Cubic interpolation between two attitudes

First consider the following problem: we are given two direction cosine matrices \( C_s \triangleq C(0) \) and \( C_f \triangleq C(T) \) and the initial and final angular velocities \( \omega_s \) and \( \omega_f \) and we wish to find an interpolant \( C(\tau) \) consistent with the given conditions. We write the interpolant in terms of the rotation vector \( \theta(\tau) \) as follows:

\[
C(\tau) = C_s \exp (\theta(\tau) \times) \tag{1}
\]

The rotation vector is parameterized as a cubic function:

\[
\theta = a \tau + b \tau^2 + c \tau^3 \tag{2}
\]

where \( a, b, c \) are constant and yet unknown vectors. Relations between the rotation vector time derivative and the angular velocity are provided in [2] and they can be written as follows:

\[
\dot{\theta} = A(\theta) \omega \tag{3a}
\]

\[
A(\theta) = I + \frac{1}{2} (\theta \times) + \frac{1}{\theta^2} \left( \frac{\theta}{2} \cot \left( \frac{\theta}{2} \right) \right) (\theta \times)^2 \tag{3b}
\]

and

\[
\omega = A^{-1}(\theta) \dot{\theta} \tag{4a}
\]

\[
A^{-1}(\theta) = I - \frac{1 - \cos \theta}{\theta^2} (\theta \times) + \frac{\theta - \sin \theta}{\theta^3} (\theta \times)^2 \tag{4b}
\]

The conditions which were formulated are expressed formally as:

\[
\theta(T) = \Theta \tag{5a}
\]

\[
\dot{\theta}(0) = \omega_s \tag{5b}
\]

\[
\dot{\theta}(T) = \dot{\Theta} \tag{5c}
\]

with

\[
\Theta = \log C_s^T C_f \tag{5d}
\]

\[
\dot{\Theta} = A(\Theta) \omega_f \tag{5e}
\]

Here \( \Theta \) is the total rotation vector and \( \log \) is an operation to extract a rotation vector from a rotation matrix.

Substituting (2) into (5) we get the following system of equations for the coefficients:

\[
aT + T^2 b + T^3 c = \Theta \tag{6a}
\]

\[
a = \omega_s \tag{6b}
\]

\[
a + 2Tb + 3T^2 c = \dot{\Theta} \tag{6c}
\]
And the solution to it:

\[ a = \omega_0 \]  
\[ b = (3\Theta_f - 2T\omega_0 - T\Theta)/T^2 \]  
\[ c = (-2\Theta_f + T\omega_0 + T\Theta)/T^3 \]

So the interpolant can be found easily provided that the initial and the final angular velocities are known. Let’s also write the expression for the second derivative at the start and end times:

\[ \ddot{\theta}(0) = (6\Theta - 4T\omega_0 - 2T\Theta)/T^2 \]  
\[ \ddot{\theta}(T) = (-6\Theta + 2T\omega_0 + 4T\Theta)/T^2 \]

2 Smooth interpolation of a series of attitudes

Now consider a sequence of attitude matrices \( C(t_n), n = 0,\ldots, N \). We want to find interpolants \( \theta_n(\tau), 0 \leq \tau \leq T_n \approx t_n - t_{n-1} \) between them. If we somehow determine angular rates \( \omega_n = \omega(t_n) \) then we will be able to compute the cubic interpolants as described in the previous section. In order to find these angular rate we require the continuity of the angular \( \omega \) rate and acceleration \( \beta = \dot{\omega} \) at all \( t_n, n = 1,\ldots, N - 1 \).

We find the angular acceleration by differentiating (4):

\[ \beta(\theta) = A^{-1}(\theta)\ddot{\theta} + \dot{A}^{-1}(\theta)\dot{\theta} \]  

The second term is a quadratic function of \( \dot{\theta} \) and it’s zero when \( \theta = 0 \). Let’s denote it as:

\[ \dot{A}^{-1}(\theta)\ddot{\theta} = \Delta\beta(\theta, \dot{\theta}) = -\theta\sin\theta + 2(\cos\theta - 1) \frac{\theta + \theta\cos\theta - 3\sin\theta}{\theta^3} (\theta \cdot (\theta \times \dot{\theta})) + \frac{\theta - \sin\theta}{\theta^3} (\dot{\theta} \times (\theta \times \dot{\theta})) \]  

The condition of the angular acceleration continuity at time \( t_n \) reads:

\[ A^{-1}(\Theta_n)\ddot{\theta}(T_n) + \Delta\beta(\Theta_n, \dot{\theta}_n) = \ddot{\theta}_{n+1}(0) \]  

where

\[ \Theta_n = \log C^T_{n-1}C_n \]  
\[ \dot{\Theta}_n = A(\Theta_n)\omega_n \]

Substituting [3] into (11) we get:

\[ \frac{2}{T_n} A_n^{-1}\omega_{n-1} + \left( \frac{4}{T_n} + \frac{4}{T_{n+1}} \right) \omega_n + \frac{2}{T_{n+1}} A_{n+1}\omega_{n+1} = \frac{6}{T^2_n} \Theta_n + \frac{6}{T^2_{n+1}} \Theta_{n+1} - \Delta\beta_n \]

where

\[ A_n \doteq A(\Theta_n) \]  
\[ A_n^{-1} \doteq A^{-1}(\Theta_n) \]  
\[ \Delta\beta_n \doteq \Delta\beta(\Theta_n, \dot{\theta}_n) \]

Here we also considered that \( A^{-1}(\Theta_n)\Theta_n = \Theta_n \). Two additional boundary conditions are required, we choose to simply set:

\[ \omega_0 = \frac{1}{T_0} \Theta_1 \]  
\[ \omega_N = \frac{1}{T_N} \Theta_N \]
But other choices are also possible.

Without $\Delta \beta_n$, the system (12) would be linear and very similar to one arising in the cubic spline interpolation. In order to solve this system we can compute consecutive approximations $\omega_n^{(i)}$ by solving the linear system with known $\Delta \beta_n^{(i)} = \Delta \beta(\Theta_n, A_n \omega_n^{(i)})$. With fixed $\Delta \beta_n^{(i)}$ the system is block tridiagonal and can be solved either by a two-pass algorithm considering the block structure or it could be represented as a banded matrix (with bandwidth equal to 11) and solved by a library solver. The convergence is declared if:

$$\left| \omega_n^{(i+1)} - \omega_n^{(i)} \right| < \text{tol} \left( 1 + \left| \omega_n^{(i+1)} \right| \right)$$

(14)

The inequality is checked element-wise. For the initial approximation we use:

$$\omega_n^{(0)} = \frac{1}{T_n} \Theta_n.$$  

(15)

Although there is no proof provided here, but the convergence is confirmed by practice: about 5 iterations are typically necessary with $\text{tol} = 10^{-9}$.

2.1 Example of interpolation

We consider the attitude sequence represented by the Euler angles in Table 1. The results of the application of

<table>
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<th>Time (s)</th>
<th>Heading (deg)</th>
<th>Pitch (deg)</th>
<th>Roll (deg)</th>
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<td>0</td>
<td>0</td>
</tr>
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<td>30</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Attitude sequence

the proposed interpolation algorithm are shown on Fig. 1. We see reasonable smooth curves for interpolated Euler angles with typical “overshoots” inherent for the cubic spline interpolation. The angular velocity and acceleration are continuous as expected, whereas the jerk is not. Also note that the angular acceleration is not a piecewise-linear function in contrast to the second derivative of the cubic spline. This is because the angular acceleration is not equal to the second derivative of the rotation vector, which is a piecewise-linear function.

References


Figure 1: Interpolation results