Differential geometric algebra using Leibniz, Grassmann

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Abstract

The Grassmann.jl package provides tools for computations based on multi-linear algebra and spin groups using the extended geometric algebra known as Grassmann-Clifford-Hestenes-Leibniz algebra. The primary operations are outer, regressive, inner, geometric, and cross products along with the Hodge star, adjoint, and multivector reversal operations. The kernelized operations are built on composite sparse tensor products and Hodge duality, with high dimensional support for up to 62 indices and staged caching / precompilation. Code generation enables making concise and extensible definitions. The DirectSum.jl multivector parametric type polymorphism is based on various tangent vector spaces and conformal projective geometry. Additionally, interoperability between different sub-algebras is enabled by AbstractTensors.jl, on which the type system is built.

In this paper, the mathematical foundations and some of the nuances in the definitions specific to the Grassmann.jl implementation are concisely described. The Grassmann.jl package and its accompanying support packages provide an extensible platform for computing with geometric algebra at high dimensions. The design is based on the TensorAlgebra abstract type system interoperability from AbstractTensors.jl with a VectorBundle type parameter from DirectSum.jl. Abstract vector space type operations happen at compile-time, resulting in a differential conformal geometric algebra of hyper-dual multivector forms.

The nature of the geometric algebra code generation enables one to easily extend the abstract product operations to any specific number field type (including Leibniz differentials with Leibniz.jl or symbolic coefficients with Reduce.jl), by taking advantage of Julia’s type system. With the type system, it is possible to construct mixed tensor products with their coefficients, e.g. bivector elements of Lie groups [6][9].

1 Direct sum parametric type polymorphism

The DirectSum.jl package is a work in progress providing the necessary tools to work with arbitrary dual VectorBundle elements with optional origin. Due to the parametric type system for the generating VectorBundle, the Julia compiler can fully preallocate and often cache values efficiently ahead of run-time. Although intended for use with the Grassmann.jl package, DirectSum can be used independently.

Definition 1 (Vector bundle of submanifold). Let $T^\mu V \in \text{Vect}_k$ be a VectorBundle<:Manifold of rank $n$, $T^\mu V = (n, P, g, \mu), \quad n, \mu \in \mathbb{N}, \quad P \subseteq \langle v_\infty, v_0 \rangle, \quad g : V \times V \to k$.

The type VectorBundle$(n, P, g, \mu)$ uses byte-encoded data available at pre-compilation, where $P$ specifies the null-basis from the projective split, $g$ is a bilinear form that specifies the metric of the space, and $\mu$ is an integer specifying the order of the tangent bundle (i.e. multiplicity limit of Leibniz-Taylor monomials).
The dual space functor \( (\cdot)' : \text{Vect}_k^p \rightarrow \text{Vect}_k \) is an involution which toggles a dual vector space with inverted signature with property \( V' = \text{Hom}(V, k) \) and having the Basis generators
\[
\langle v_1, \ldots, v_n, \partial_1, \ldots, \partial_n \rangle = T^{\mu}V \leftrightarrow T^{\mu}V' = \langle w_1, \ldots, w_n, \epsilon_1, \ldots, \epsilon_n \rangle.
\]

The metric signature of the Basis(V,1) elements of a vector space \( V \) can be specified with the \( \psi^{*\ldots*} \) constructor by using + and - to specify whether the Basis(V,1) element of the corresponding index squares to +1 or -1. For example, \( \psi^{+++*} \) constructs a positive definite 3-dimensional VectorBundle. It is possible to specify an arbitrary DiagonalForm for the basis elements with \( \psi^{0,0,0} \) or \( \psi^{-1,1,1,*} \), although the \( \pm \) format has better performance. Further development of the DirectSum.jl will result in more metric types.

The direct sum operator \( \oplus \) can be used to join spaces (alternatively \( + \)). The direct sum of a VectorBundle and its dual \( \vee V \) represents the full mother space \( \psi^* \). Additionally to the direct-sum operation, several other operations are supported, such as \( \cup, \cap, \subseteq, \supseteq \)

\[ V \oplus V' \]

and its dual
\[ V^* \oplus V'^* \]

The direct sum operator \( \oplus \) is intended for universal interoperability of the abstract TensorAlgebra type system. All TensorAlgebra(V) subtypes contain \( V \) in their type parameters, used to store a VectorBundle value obtained from the DirectSum.jl package. By itself, this package does not impose any structure or specifications on the TensorAlgebra(V) subtypes and elements, aside from requiring \( V \) to be a VectorBundle. This means that different packages can create tensor types with a common underlying VectorBundle structure.

**Definition 2.** Let \( V \in \text{Vect}_k \) be a VectorBundle with dual space \( V' \) and the basis elements \( w_k : V \rightarrow k \), then for all \( x \in V, c \in k \) the properties \( (w_i + w_j)(x) = w_i(x) + w_j(x) \) and \( (cw_i)(x) = cw_i(x) \) hold. An element of a mixed-symmetry TensorAlgebra(V) is a multilinear mapping formally constructed by taking the tensor products of linear and multilinear maps, \( (\otimes_k \omega_k)(v_1, \ldots, v_{\sum p_k}) = \prod_k \omega_k(v_1, \ldots, v_{p_k}) \).

**Definition 3** (Mixed-symmetry basis). Combining linear basis generating elements with each other using the multi-linear tensor product yields a graded (decomposable) tensor Basis \( \langle w_{p_1} \otimes \ldots \otimes w_{p_k} \rangle_k \), where the grade \( k \) is determined by the number of basis elements in the tensor product decomposition. The algebra is partitioned into symmetric and anti-symmetric tensor equivalence classes. For any pair of tensors, either
\[
\omega \otimes \eta = -\eta \otimes \omega, \quad \text{anti-symmetric}
\]
\[
\omega \otimes \eta = \eta \otimes \omega, \quad \text{symmetric}
\]

Typically the \( k \) in \( (\partial_{p_1} \otimes \ldots \otimes \partial_{p_k})^{(k)} \) is referred to as the order of the element for fully symmetric tensors, which is tracked separately from the grade such that \( \partial_k \langle w_j \rangle_r = \langle \partial_k w_j \rangle_r \) and \( \partial_k \langle \partial_k \rangle^{(r)} \omega_j = \langle \partial_k w_j \rangle^{(r)} \). Hence, there is a partitioning into even grade components \( \omega_+ \) and odd grade components \( \omega_- \) with \( \omega_+ + \omega_- = \omega \).

**Remark.** Observe that the anti-symmetric property implies that \( \omega \otimes \omega = 0 \), while the symmetric property neither implies nor denies such a property. In 1862, Grassmann remarked [5] that the symmetric algebra of functions is by far more complicated than his anti-symmetric exterior algebra. The first part of the book focused on anti-symmetric exterior algebra, while the more complex symmetric function algebra of Leibniz was subject of the second part of the book. Elements \( \omega_k \) in the space \( \Lambda V \) of anti-symmetric algebra are often studied as unit quantum state vectors in a unitary probability space, where \( \sum_k \omega_k \neq \otimes_k \omega_k \) is entanglement.
**Definition 4.** The Grassmann anti-symmetric exterior basis is denoted by \( v_{i_1 \ldots i_p} \in \Lambda^p V \) with its dual basis \( w^{i_1 \ldots i_p} \in \Lambda^q V \), while the Leibniz symmetric basis will be \( \partial_{i_1} \ldots \partial_{i_q} \in L^q V \) with \( \epsilon^{i_1 \ldots i_q} \in L^q V \) dual elements.

Higher-order composite tensor elements will be denoted by an oriented-multi-set \( X \in \text{OMSet} \) such that \( w_X = \bigotimes_k w_{i_k}^{\mu_k} \) with \( X = \{ (i_1, \mu_1), \ldots, (i_\mu, \mu_\mu) \} \) and \( |X| = \sum_k \mu_k \) is grade+order. Indices \( AX \subseteq AV \) that are anti-symmetric have two orientations and higher multiplicities of those result in zero values, so the only interesting multiplicity is \( \mu_k \equiv 1 \). Symmetric tensors have an ambiguous multiplicity of nilpotence; decide that \( \epsilon_k^{\mu_k+1} = 0 \), so \( \mu_k \leq \mu \) can be non-trivial, negative, or possibly unbounded. Grassmann’s exterior algebra is fundamentally simpler in structure than the symmetric generated algebra, as it doesn’t invoke the properties of multi-sets. The exterior Grassmann index algebra is related to the algebra of oriented sets and the Leibniz symmetric algebra is that of unoriented multi-sets. Combined, the mixed-symmetry algebra yield a multi-linear proposition lattice. The formal sum of equal grade elements is an oriented Chain and with mixed grade it is a MultiVector. Thus, various standard operations on the oriented multi-sets are possible including \( \cup, \cap, \ominus \) and most importantly the \( X \ominus Y = (X \cup Y) \setminus (X \cap Y) \) symmetric difference operation \( \vee \).

In order to work with a TensorAlgebra\((V)\), it is necessary for some computations to be cached. This is usually done automatically when accessed. The staging of the precompilation and caching is designed so that a user can smoothly transition between very high dimensional and low dimensional algebras in a single session, with varying levels of extra caching and optimizations. The parametric type formalism in Grassmann.Algebra\((V)\) is a container for the TensorAlgebra generators of \( V \), the Grassmann.Algebra is only cached for \( n \leq 8 \). For a VectorBundle\((n)\) of dimension \( 8 < n \leq 22 \), the Grassmann.SpaseAlgebra type is used. To reach higher dimensions, for \( n > 22 \) the Grassmann.ExtendedAlgebra type is used.

### 3 Geometric algebraic product structure

While oriented sets of the Grassmann exterior algebra are simpler, the parity \([1] \) of \((-1)^\Pi \) is factored into transposition compositions when interchanging the ordering of the tensor product argument permutation. Alternatively, symmetric algebra does not need to track the parity but does have multiplicity in its indices. Symmetric differential function algebra of Leibniz trivializes the orientation property of index multi-sets, while Grassmann’s exterior algebra is partitioned into two oriented equivalence classes by anti-symmetry. The full tensor algebra can be sub-partitioned into equivalence classes in multiple ways based on the composite symmetry, grade, and metric signature properties. Both of the symmetry classes can be characterized by the same geometric product, which is written as multiplication but explicitly denoted by \( \ominus \) for clarity here.

**Definition 5.** The geometric algebraic product is the \( \Pi \) oriented symmetric difference operator \( \ominus \) (weighted by the bilinear form \( g \)) and multi-set sum \( \ominus \) applied to multilinear tensor products \( \otimes \) in a single operation:

\[
\omega_X \ominus \eta_Y = (-1)^{\Pi(AX,AY)} \det[g_{\Lambda(X \cap Y)}] \left( \bigotimes_{k \in \Lambda(X \cap Y)} w_{i_k}^{\mu_k} \right) \ominus_k \left( \bigotimes_{k \in L(X \cap Y)} \epsilon_{i_k}^{\mu_k} \right)
\]

\( \Lambda^\Lambda \)-anti-symmetric, \( \Lambda^g \)-mixed-symmetry, \( L^g \)-symmetric

*Note.* The product symbol \( \ominus \) will be used to explicitly denote usage of the geometric algebraic product, although the standard number product \( \ast \) notation could also be used. This choice is to help emphasize that the geometric algebraic product is characterized by symmetric differencing of anti-symmetric indices.

**Definition 6.** The symmetry properties of the tensor algebra can be characterized in terms of the geometric product by two averaging operations, which are the symmetrization \( \bigotimes \) and anti-symmetrization \( \bigodot \) operators:

\[
\bigotimes_{k=1}^j \omega_k = \frac{1}{j!} \sum_{\sigma \in S_p} \bigotimes_k \omega_{\sigma(k)}, \quad \bigodot_{k=1}^j \omega_k = \frac{1}{j!} \sum_{\sigma \in S_p} (-1)^{\Pi(\sigma)} \bigodot_k \omega_{\sigma(k)}
\]
These products satisfy various MultiVector properties, including the associative and distributive laws.

**Definition 7** (Exterior product). Let \( w_k : V^p \times V^q \rightarrow K \) such that \( k \in \Lambda^p V \), then for all \( \sigma \in S_p \)

\[
(-1)^{\Pi(\sigma)} \left( \bigotimes_k \omega_k(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \right) \sim \bigwedge_k \omega_k(v_1, \ldots, v_p) \iff \bigotimes_k \omega_k = \bigwedge_k \omega_k
\]

there is an equivalence relation \( \sim \) which holds. It has become typical to use the \( \wedge \) product symbol to denote products of such elements as \( \Lambda V \equiv \bigotimes \Lambda V / \sim \) modulo anti-symmetrization.

**Definition 8** (Symmetric tensor product of Leibniz differentials). Let \( \partial_k = \frac{\partial}{\partial x_k} \in L_0 V \) Leibnizian symmetric tensors, then there is an equivalence relation \( \sim \) which holds for each \( \sigma \in S_p \) along with each derivation property,

\[
(\bigotimes_k \partial_{\sigma(k)}) \omega \sim (\partial_{p \circ \cdots \circ 1}) \omega \iff \bigotimes_k \partial_k = \bigotimes_k \partial_k,
\]

\( \partial_k(\omega \eta) = \partial_k(\omega) \eta + \omega \partial_k(\eta) \).

Since VectorBundle choices are fundamental to TensorAlgebra operations, the universal interoperability between TensorAlgebra(V) elements with different associated VectorBundle choices is naturally realized by applying the union morphism to type operations. For example, \( \Lambda : \Lambda^p V_1 \times \cdots \times \Lambda^p V_g \rightarrow \Lambda^p \bigcup_k V_k \).

**Definition 9** (Reverse, involute, conjugate). The reverse of \( \langle \omega \rangle_r \) is defined as \( \langle \tilde{\omega} \rangle_r = (-1)^{(r-1)r/2} \langle \omega \rangle_r \), while the involute is \( (-1)^r \langle \omega \rangle_r \) and Clifford conj \( \langle \omega \rangle^\dagger \) is the composition of involute and reverse.

**Definition 10** (Reversed product). The reversed geometric product \( * \) yields a Hilbert space structure:

\[
\omega * \eta = \tilde{\omega} \Theta \eta, \quad \omega *' \eta = \omega \Theta \tilde{\eta}, \quad |\omega|^2 = \omega * \omega, \quad |\omega| = \sqrt{\omega * \omega}, \quad ||\omega|| = \text{Euclidean } |\omega|
\]

**Definition 11** (Sandwich product). The sandwich product is defined as \( \eta \odot \omega = (\eta \ast \omega) \odot \eta \). Alternatively, the reversed definition is \( \eta \odot \omega = \eta \ast (\omega *' \eta) \) or in Julia \( \texttt{η} \gg \gg \texttt{ω} \), which is often found in literature.

*Note.* Observe that \( * \) and \( *' \) could both be used in abs, abs2, and norm; however they are different products. The scalar product \( \Theta \) is the scalar part of the reversed product \( \eta \ast \omega = \langle \eta \ast \omega \rangle \). In general \( \sqrt{\omega} = e^{(\log \omega)/2} \) is valid for invertible multivectors \( \omega^{-1} = \omega * |\omega|^{-2} = \tilde{\omega}/|\omega|^2 \), where \( \eta/\omega = \eta \odot \omega^{-1} = \eta \odot (\tilde{\omega}/(\omega \odot 1)) \).

*Remark.* It is overall more simple to use the \( * \) and \( \odot \) operations instead of the \( *' \) and \( \Theta \) variations.

The real part \( \Re \omega = (\omega + \tilde{\omega})/2 \) is defined by the property \( |\Re \omega|^2 = (\Re \omega)^{\odot 2} \) and the imaginary part \( \Im \omega = (\omega - \tilde{\omega})/2 \) by \( |\Im \omega|^2 = -(\Im \omega)^{\odot 2} \), such that for any multivector \( \omega = \Re \omega + \Im \omega \) has real and imaginary partitioned by

\[
\langle \omega \rangle_r = \sqrt{\langle \tilde{\omega} \rangle_r^2 / \langle \omega \rangle^2_r} = \sqrt{\langle \tilde{\omega} \rangle_r \ast \langle \omega \rangle^{-1}_r} = \sqrt{\langle \tilde{\omega} \rangle_r / \langle \omega \rangle_r} = \sqrt{(-1)^{(r-1)r/2} \in \{1, \sqrt{-1}\}},
\]

which is a unique partitioning completely independent of the metric space and manifold of the algebra \([7]\).

\[
\omega * \omega = |\omega|^2 = |\Re \omega + \Im \omega|^2 = |\Re \omega|^2 + |\Im \omega|^2 + \Re \omega * \Im \omega + \Im \omega * \Re \omega = |\Re \omega|^2 + |\Im \omega|^2 + 2\Re(\Re \omega * \Im \omega)
\]

The radial and angular components in a multivector exponential are partitioned by the parity of their metric.

The Grassmann Basis elements \( v_k \in \Lambda^1 V \) and \( w_k \in \Lambda^1 V \) are linearly independent vector and covector elements of \( V \), while the Leibniz Operator elements \( \partial_k \in \Lambda^1 V \) are partial tangent derivations and \( \epsilon_k(x) \in \Lambda^1 V \) are dependent functions of the tangent manifold. Higher grade elements of \( \Lambda V \) correspond to SubManifold subspaces, while higher order function elements of \( LV \) become homogenous polynomials and Taylor series.

A universal unit volume element can be specified in terms of LinearAlgebra.UniformScaling, which is independent of \( V \) and has its interpretation only instantiated by the context of the TensorAlgebra(V) element being operated on. The universal interoperability of LinearAlgebra.UniformScaling as a pseudoscalar element which takes on the VectorBundle form of any other TensorAlgebra element is handled globally. This enables the usage of \( I \) from LinearAlgebra as a universal pseudoscalar element defined at every point \( x \) of the manifold, which is mathematically denoted by \( I = I(x) \) and specified by the \( g(x) \) bilinear tensor field of \( TM \).

**Definition 12** (Grassmann-Poincare-Hodge complement). Let \( * \langle \omega \rangle_p = \langle \omega \rangle_p * I \), then \( * : \Lambda^p V \rightarrow \Lambda^{n-p} V \).

*Remark.* While \( * \omega \) is complementright of \( \omega \), the complementleft would be \( I * \omega \). The \( * \) symbol was added to the Julia language as unary operator for ease of use with Grassmann on Julia’s v1.2 release.
4 Leibniz operators and Grassmann’s Hodge-DeRham theory

John Browne has discussed the Grassmann duality principle in book [3], stating that every theorem (involving exterior and regressive products) can be translated into its dual theorem by replacing the ∧ and ∨ operations and applying Poincare duality (homology). First applying this Grassmann duality principle to the ∧ product alone, let \( \{ \omega_k \}_k \in \Lambda^p V, P = \sum_k p_k \), then the co-product \( \vee : \Lambda^p_1 V_1 \times \ldots \times \Lambda^p_s V_s \rightarrow \Lambda^{p-s+1} V \), is obtained. in Grassmann’s original notation ∧, ∨, * operations were combined. The join ∧ product is analogous to union ∪, the meet ∨ product is analogous to intersection ∩, and the orthogonal complement * \( \rightarrow \) is negation. Together, \( (\land, \lor, * ) \) yield an orthocomplementary propositional lattice (quantum logic) by

\[
(\ast \sum_k \omega_k)(v_1, \ldots, v_P) = (\sum_k \ast \omega_k)(v_1, \ldots, v_P) \quad \text{DeMorgan’s Law.}
\]

**Definition 13.** The left \( \ast \) and right \( \ast \) contraction symmetrically define \( \langle \omega \rangle_{\ast_r}, \langle \eta \rangle_{\ast_l} = \begin{cases} 
\omega \eta = \omega \vee \ast \eta & r \geq s \\
\omega \eta = \eta \vee \ast \omega & r \leq s
\end{cases}.
\]

Note that for \( \omega, \eta \) of equal grade, the operations \( \omega \circ \eta = \omega \land \eta = \omega \cdot \omega \neq \omega \eta \) are symmetric.

**Definition 14.** Let \( \nabla = \sum_k \partial_k v_k \) and \( \epsilon = \sum_k \epsilon_k(x) v_k \in \Omega^1 V \) be unit sums of the mixed-symmetry basis. Elements of \( \Omega^p V \) are known as differential p-forms and both \( \nabla \) and \( \epsilon \) are tensor fields dependent on \( x \in W \). Another notation for a differential form is \( dx_k = \epsilon_k(x) w_k \), such that \( \epsilon = dx_k/x_k \) and \( \partial_k w(x) = \omega' \). 

**Note.** The space \( W \) does not have to equal \( V \in \text{Vect}_k \), above, as \( \Omega^p V \) could have coefficients from \( K = LW \).

**Definition 15.** Define [2] the differential \( d : \Omega^p V \rightarrow \Omega^{p+1} V \) and co-differential \( \delta : \Omega^p V \rightarrow \Omega^{p-1} V \) such that

\[
\ast d\omega = \ast (\nabla \land \omega) = \nabla \land \omega, \quad \omega \land \nabla = \omega \lor \ast \nabla = \partial \omega = -\delta \omega.
\]

These two maps have the special properties \( d \circ d = 0 \) and \( \partial \circ \partial = 0 \) for any form \( \omega \) and tensor field \( \nabla \). In topology there is boundary operator \( \partial \) that can be defined by \( \partial \epsilon = \epsilon \cdot \nabla = \sum_k \partial_k \epsilon_k \) and is commonly discussed in terms the limit \( (\epsilon(x) \cdot \nabla)(x) = \lim_{h \to 0} \frac{\omega(x+hx) - \omega(x)}{h} \), which is the directional derivative [10].

**Example 1** (Vorticity of vector-field). \( *d(dx_1 + dx_2 + dx_3) = (\partial_2 - \partial_3) dx_1 + (\partial_3 - \partial_1) dx_2 + (\partial_1 - \partial_2) dx_3 \).

**Example 2** (Boundary of 3-simplex). Faces as oriented Chain: \( \partial(w_{1234}) = -\partial_4 w_{123} + \partial_3 w_{124} - \partial_2 w_{134} + \partial_1 w_{234} \).

**Theorem 1** (Leibniz-Taylor series). Let \( \partial X = \biguplus_k \partial_k^k \) be defined so that \( |X| = \sum_k \mu_k \), then \( e^{\partial \epsilon} \omega(x) \) is

\[
\sum_{\mu = 0}^\infty \frac{(\partial \epsilon)^{\partial \mu}}{\partial \mu!} \omega(x) = \sum_{\mu = 0}^\infty \frac{\sum_k \partial_k \epsilon_k(x)^{\partial \mu}}{\partial \mu!} \omega(x) + (\mu + 1) \sum_{|X| = \mu + 1} \int_0^1 (1 - t) \sum_k \partial_k \epsilon_k(x)^{\partial \mu} \omega(x + te) \, dt.
\]

The multivariate chain rule is encoded into the geometric algebraic product when using mixed-symmetry.

**Theorem 2** (Hilbert adjoint Hodge-DeRham operators). Let \( \nabla \in \Omega^1 V \) be a Leibnizian vector field operator, then \( \partial, -\partial \) are Hilbert adjoint Hodge-DeRham operators with \( \langle * \rangle \)

\[
\int_M d\omega \land * \eta + \int_M \omega \land * \partial \eta = 0, \quad \langle d\omega \land * \eta \rangle = \langle \omega \land * -\partial \eta \rangle
\]

**Proof.** Observe that \( \partial \omega = -\omega \land \nabla = \ast (\ast \omega + \ast^2 \nabla) = (-1)^n (-1)^{n-k} * d \ast \omega \). Then substitute this into the integral \( \int_M \omega \land (-1)^m k_m+1 \ast \ast \partial \ast \eta = (-1)^{k_m+1} (-1)^{n-k} - (-1)^{k_m+1} (-1)^{n-k+1} \int_M \omega \land \partial \ast \eta, \) with \( (-1)^{k_m+1} (-1)^{n-k+1} \) \( (-1)^{k_m+1} \) and also \( (-1)^n \int_M \omega \land d \ast \eta = \int_M d(\omega \land * \eta) - (-1)^{k_m+1} \omega \land d \ast \eta = \int_M d\omega \land * \eta. \) The other identity can be proved by relying on a variant of the common factor theorem by Browne [3].

**Theorem 3** (Clifford-Dirac-Laplacian). Dirac operator \( \Delta^\frac{1}{2} \omega = \pm \omega \land \nabla = \pm \nabla \land \omega \pm \omega \land \nabla = \pm d \omega \pm \partial \omega \) \[4\].

\[
\Delta \omega = \nabla \land (\omega \land \nabla) + (\nabla \land \omega) \land \nabla) = \pm (\pm \omega \land \nabla) \land \nabla), \quad \mathcal{H}^p M = \{ \Delta \omega = 0 : \omega \in \Omega^p M \}
\]

Elements \( \omega \) are harmonic if \( \Delta \omega = 0 \) and both exact \( d\omega = 0 \) and coexact \( \partial \omega = 0 \), Hodge decomposition \[11\]: \( \Omega^p M = \mathcal{H}^p M \oplus \im(d\Omega^{p-1} M) \oplus \im(\partial \Omega^{p+1} M) \). Warning \( \nabla \omega \neq \omega \nabla, \nabla^2 \omega = \omega \nabla^2 \) for higher-order tensor fields!
Definition 16 (Null-basis of projective split). Let $v^2 = \pm 1$ be a basis with $v_{\infty} = v_+ + v_-$ and $v_0 = (v_+ - v_-)/2$.

An embedding space $\mathbb{R}^{n+1,1}$ carrying the action from the group $SO(n+1,1)$ then has $v_{\infty}^2 = 0$, $v_0^2 = 0$, $v_{\infty} \cdot v_0 = 1$, and $v_{\infty}^2 = 1$ with Minkowski plane $v_{\infty \emptyset}$ having the Hestenes-Dirac-Clifford product properties

\[ v_{\infty \emptyset} \cdot v_{\emptyset \emptyset} = -v_{\infty \emptyset}, \quad v_{\infty \emptyset} \otimes v_0 = v_0, \quad v_{\infty \emptyset} \otimes v_0 = -1 + v_{\infty \emptyset}, \quad v_0 \otimes v_{\infty \emptyset} = -1 - v_{\infty \emptyset} \]

Declaring an additional null-basis is done by specifying it in the string constructor with $\emptyset$ at the first index (i.e. $S^{\emptyset \emptyset ++}$). Likewise, an optional origin can be declared by $\emptyset$ subsequently (i.e. $S^{\emptyset \emptyset +++}$ or $S^{\emptyset \emptyset +++}$). These two basis elements will be interpreted in the type system such that they propagate under transformations when combining a mixed index sets (provided the Signature is compatible).

Example 3 (using Leibniz $\boxplus$ Grassmann).\[ \texttt{julia> S}^{\emptyset \emptyset +++}(\nabla)^2 \mapsto \Delta = (-2\partial_{\infty \emptyset} + \partial_1^2 + \partial_2^2 + \partial_3^2)\hat{v} \]

5 Conclusion

Grassmann.jl and its accompanying support packages provide an extensible platform for fully general computing with geometric algebra at high dimensions. All of the types and operations in this paper are implemented using only a few thousand lines of code with Julia’s type polymorphism code generation, although the mixed-symmetry interaction of Leibniz.jl and Grassmann.jl is still a work in progress. The goal is to algebraically define operator actions on groups of Leibniz.jl differential function elements. One way is to use Reduce.jl for differentiation and SyntaxTree.jl to obtain optimal symbolic polynomial forms [8].

Thus, these packages provide general rotational algebras and Lie bivector groups with a full trigonometric suite enabled by $e^A = \sum_n \frac{A^n}{n!}$. Conformal geometric algebra is possible with the Minkowski plane $v_{\infty \emptyset}$, based on the null-basis. In general, multivalued quantum logic is enabled by the \wedge, \vee, * Grassmann-Hodge dual lattice. Mixed-symmetry algebra with Leibniz.jl and Grassmann.jl, having the geometric algebraic product chain rule, yields automatic differentiation and Hodge-DeRahm co/homology as unveiled by Grassmann.

\[
\begin{align*}
0 \frac{d}{d^\Omega} \Omega^0(M) \frac{d}{d^\Omega} \Omega^1(M) \frac{d}{d^\Omega} \cdots \frac{d}{d^\Omega} \Omega^n(M) \frac{d}{d^\Omega} 0, \quad \mathcal{H}^p M \cong \frac{\ker(d\Omega^p M)}{\dim(d\Omega^{p-1} M)}, \quad \dim \mathcal{H}^p M = \dim \frac{\ker(d\Omega^p M)}{\dim(d\Omega^{p+1} M)}
\end{align*}
\]

In conclusion, the Dirac-Clifford product yields generalized Hodge-Laplacian and $b_p = \dim \mathcal{H}^p M$ are the Betti numbers with Euler characteristic $\chi = \sum_p (-1)^p b_p$. There will be a more detailed follow-up paper.

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References